Odyssey Research Programme

Title: Metric Spaces and Isometries
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Motivation

This poster focuses on metric spaces and a related theorem that might possibly be refined to a statement of equivalence.

Definitions

Definition 1 . (Metric Space) A **metric space** is a pair (X, d) where X is a set and d is a metric on X (distance function) that is a function defined on $X \times X$ such that for all $x, y, z \in X$, we have:

- $d(x, y) \geq 0$
- $d(x, y) = 0$ if and only if $x = y$
- $d(x, y) = d(y, x)$ (Symmetry)
- $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality)

This definition generalizes the Euclidean metric on Euclidean space. Curious examples include $C([0, 1], \mathbb{R})$, the space of all real-valued continuous functions on $[0, 1]$ with corresponding metric $d(x, y) = \max_{t\in[0,1]} |x(t) - y(t)|$ and the space of ordered triples of zeroes and ones where the metric $d(x, y)$, known as Hamming distance is the number of places where x and y have different entries.

Definition 2 . (Isometry) Let $X = (X, d)$ and $\tilde{x} = (\tilde{X}, \tilde{d})$ be two metric spaces. Then a mapping $T : X \to \tilde{T}$ is said to be an **isometry** [1] if T preserves distance, that is, for all $x, y \in X$, $d(x, y) = \tilde{d}(Tx, Ty)$. A space X is said to be **isometric** with $\tilde{X}$ if there exists a bijective isometry of X onto $\tilde{X}$.

It is easy to see from this definition that an isometry must be an injective (one-to-one). Thus, surjective (onto) isometry is equivalent to bijective isometry. A bijection is a mapping that is one-to-one and onto. Clearly, every metric space is isometric to itself via identity mapping. Concrete examples of bijective isometries include translation and rotation in Euclidean space.

Definition 3 .(Group) A **group** is a set G equipped with a binary operation $\circ$, $(G, \circ)$ such that:

- $x \circ y \in G$ for each $x, y \in G$ (Closure)
- $x \circ (y \circ z) = (x \circ y) \circ z$ for each $x, y, z \in G$ (Associativity)
- there exists an element e such that $x \circ e = x = e \circ x$ for each $x \in G$ (Identity)
- for each $x \in G$, there exists $x^* \in G$ such that $x \circ x^* = e = x \circ x^*$ (Inverse)

The uniqueness of identity element is immediate and for each $x \in G$, the corresponding inverse element is unique. Common examples of groups include the additive group of integers, $(\mathbb{Z}, +)$ and the general linear group over $\mathbb{R}$, i.e. the group of all invertible $n \times n$ matrices of real numbers with matrix multiplication as the group operation, denoted $GL(n, \mathbb{R})$. In view of the previous definition, the set of all bijective isometries on a metric space X, denoted $Isom(X, X)$, is a group with composition of isometries as the operation. It is a subgroup of the permutation group on X (each permutation on X can be thought of as a bijection).

Definition 4 .(Group Homomorphism, Isomorphism) Let $(G_1, \circ)$ and $(G_2, \cdot)$ be two groups. A group **homomorphism** is a mapping $\phi : (G_1, \circ) \to (G_2, \cdot)$ such that for every $a, b \in G_1$, $\phi(a \circ b) = \phi(a) \cdot \phi(b)$. A group **isomorphism** is a bijective group homomorphism. If $\phi : (G_1, \circ) \to (G_2, \cdot)$ is an isomorphism, then G_1 and G_2 are isomorphic, denoted $G_1 \simeq G_2$.

A homomorphism preserves algebraic structure. It is immediate that $\phi(e_{G_1}) = e_{G_2}$ (identity is mapped to identity) if ϕ is a homomorphism. An example of a group homomorphism is a $\phi : G \to \{e\}$, where $\phi(g) = e$ for each $g \in G$, ϕ is isomorphism only if G is trivial group. The additive group of integers modulo 4, $(Z/4Z, +)$ is isomorphic to the multiplicative group of fourth roots of unity, $(\{\omega \in \mathbb{C} : \omega^4 = 1\}, \times)$.

Main Theorem

Theorem: Let $X = (X, d)$ and $\tilde{X} = (\tilde{X}, \tilde{d})$ be two metric spaces. If X and $\tilde{X}$ are isometric, then $Isom(X, X) \simeq Isom(\tilde{X}, \tilde{X})$.

Proof: By assumption, there exists a bijective isometry $T : X \to \tilde{X}$. Define $\phi : Isom(X, X) \to Isom(\tilde{X}, \tilde{X})$ where $\phi(U) = TUT^{-1}$, it is easy to verify ϕ is a group homomorphism and its inverse is given by $\phi^{-1}(\tilde{U}) = T^{-1}\tilde{U}T$. Thus, ϕ is an isomorphism and hence the conclusion.

The converse does not hold. A counterexample is $X = \{1, 2, 3\} \subseteq \mathbb{N}$ and $\tilde{X} = \{1, 2\} \subseteq \mathbb{N}$, both are equipped with the induced Euclidean metric d, where $d(a, b) = |a - b|$. Clearly, $Isom(X, X) \simeq Isom(\tilde{X}, \tilde{X}) \simeq C_2$, the cyclic group of order 2 but X and $\tilde{X}$ are not isometric.

What conditions do we need to make this theorem a statement of equivalence? By Banach-Stone theorem [2], if X and $\tilde{X}$ are compact Hausdorff spaces, there exists a homeomorphism from X to $\tilde{X}$ if $C(X, \mathbb{R})$ and $C(\tilde{X}, \mathbb{R})$ are isometric. We might consider the space of continuous functions on X in subsequent research.

Counting Isometries on Hamming Space

In view of Definition 1 and 2, we are interested to find the number of (bijective) isometries on the Hamming space H_n (with Hamming distance as metric) of n-bit binary codes, which amounts to $2^n \times n!$. The reasoning is as follows: let T be such isometry, denote the code with all entries being 0 by $\tilde{0}$. Without loss of generality, the image of $\tilde{0}$ under T, $T(\tilde{0})$ can take on any of the 2^n binary codes. Let e_i be the n-binary code with the i-th entry being one and zero everywhere. Before we elaborate on, define $x \bigoplus y$ to be the sum of binary codes modulo 2 in each bit. For instance, $111 \bigoplus 011 = 100$. Now for each e_i, we have the freedom to define $T(e_i) = T(\tilde{0}) + e_{k_i}$ (mind the restriction of isometry). There are $n!$ outcomes for the images of $e_{1,2,\ldots,n}$. Now the images of the remaining codes under T are determined, by addition of some combination of e_i's ($\tilde{0}$ and all the e_i's are the building blocks). We leave it for interested readers to justify. For example, if $n = 3$, $T(000) = 110$, $T(100) = T(000) \bigoplus 010 = 100$, $T(010) = 110 \bigoplus 001 = 111$, then $T(001) = 110 \bigoplus 100 = 010$ and $T(110) = T(010) \bigoplus 010 = 101$ and so forth.

Since the set of isometries forms a group, it is natural to try to classify this group up to isomorphism [3]. It turns out $Isom(H_n, H_n) \simeq A_n \rtimes B_n$ (semidirect product)

$$A_n = \{T \in Isom(H_n, H_n) : T(\tilde{0}) = \tilde{0}\} \simeq S_n \text{ and } B_n = \{T \in Isom(H_n, H_n) : Tx = x \bigoplus T(\tilde{0}) \text{ for all } x \in H_n\} \simeq \bigoplus_{i=1,\cdots,n} (\mathbb{Z}/2\mathbb{Z})_i$$

where S_n is the permutation group on n elements and $\bigoplus_{i=1,\cdots,n}(\mathbb{Z}/2\mathbb{Z})_i$ is the direct sum of n copies of additive group $Z/2\mathbb{Z}$. Clearly A_n defines a group. We note that for a general $T \in Isom(H_n, H_n)$, $T(x \bigoplus y) = T(\tilde{0}) \bigoplus Tx \bigoplus Ty$ and for each $T \in B_n$, $T^2(\tilde{0}) = \tilde{0}$. One could show B_n is a group and $B_n = \{T \in Isom(H_n, H_n) : T^2 = id\}$ where id refers to the identity map. Moreover, B_n is a normal subgroup of $Isom(H_n, H_n)$ (A_n is not). It means that for each $U \in B_n$, $TUT^{-1} \in B_n$ for any $T \in Isom(H_n, H_n)$. Finally, it is easy to verify that $A_n \cap B_n = \{id\}$ and $Isom(H_n, H_n) = A_nB_n$ (every isometry can be written as composition of isometries from A_n and B_n). These are sufficient to give semidirect product.

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